ALGEBRA II   Lesson 0509
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1. REVIEW

Recall our definitions regarding probability. We perform some sort of "experiment", such as rolling a die. An *outcome* is result of the experiment; in this case, the outcome is an integer between one and six. The *sample space* is the set of all possible outcomes. An *event* is a subset of the sample space.

The *cardinality* of a set is the number of things in it. The cardinality of the set A is denoted $|A|$.

The *probability* of event E is defined to be

$$P(E) = \frac{|E|}{|S|}.$$

For a positive integer n, we define n factorial as the product of positive integers less than or equal to n:

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n.$$

This is the number of ways of rearranging n things.

The number of *permutations* (ordered subsets) of n things taken k at a time is

$$P(n, k) = \frac{n!}{(n-k)!}.$$

The number of *combinations* (unordered subsets) of n things taken k at a time is

$$C(n, k) = \frac{n!}{k!(n-k)!}.$$

The number of combinations is typically referred to as "n choose k", and is also written

$$\binom{n}{k} = C(n, k).$$

The *cartesian product* of the sets A and B is

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

More generally, the cartesian product of sets $A_1, \ldots, A_n$ is the set of ordered n-tuples whose i^{th} entry is from the set A_i.

2. COUNTING

2.1. **Sampling.** Suppose we have a set of n objects, and we wish to select k objects from this set. Our choice of objects is called a sample, and the act of choosing them is called sampling. There are a few different methods of sampling of interest.

- *Ordered sampling*: We keep track of the order in which the objects are selected. Thus, if we select the same objects but in a different order, this is consider a different sample.
- *Unordered sampling*: The order in which objects are select does not matter. The same set of objects, selected in a different order, would be considered the same sample.
- *With replacement*: We could select an object, note which one was selected, and then return the object so that it could potentially be selected again. That is, we could allow repetitions.
- *Without replacement*: Once we select an object, we may not select it again.

This gives four different combinations of these two ideas. Each one produces a different number of distinct samples. We look at each.

We wish to count the number of ways to select k things from a set of n things. Note that when we are counting with replacement, it is possible that $n < k$, but if we are counting without replacement, it is necessary that $k \leq n$.

2.2. Ordered Sampling with Replacement. To count the number of ways to select k things from a set of n things with order and replacement, we repetitively apply the multiplication principle. We imagine filling k slots with objects from the set. There are n choices for slot 1, and n choices for slot 2, so there are n^2 choices for the first two. There are n choices for the third, and n choices for the forth, up to n choices for the kth; we continue to multiply by n and obtain

$$\text{Ordered with Replacement Selection gives } n^k \text{ samples.}$$

Note that in this case, it is possible that $n < k$.

If A is the set of n things, this form of counting is modeled by

$$A^k = \{ \text{ ordered } k\text{-tuples from } A \quad \text{where} \quad |A^k| = n^k.$$

2.3. Ordered Sampling without Replacement. To count the number of ways to select k things from a set of n things with order but without replacement, we first select one for the first slot, and have n choices; now, however, we have one less choice for the second slot, so we multiply by $n-1$. There are $n-2$ choices for the third slot, and so forth. We do this k times, multiplying as we go, and see that

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!},$$

to obtain

$$\text{Ordered without Replacement Selection gives } \frac{n!}{(n-k)!} \text{ samples.}$$

An ordered list of distinct objects is called a *permutation*. The number of permutations of k objects from a set of n objects is denoted $P(n,k)$.

If A is a set of n things, this form of counting is modeled by

$$\{ A_p^k = \text{ ordered } k \text{ tuples from } A \text{ with distinct entries } \} \quad \text{where} \quad |A_p^k| = \frac{n!}{(n-k)!}.$$

2.4. Unordered Sampling without Replacement. Here, we use the last result. Consider an unordered sample of size k, chosen without duplicates from a set of n objects. There are $k!$ ways to arrangement this sample into an ordered list, which implies that in our above computation of order sample without replacement, each such set was counted $k!$ times. Thus, we divide the number of ordered samples without replacement by $k!$, to obtain

$$\text{Unordered without Replacement Selection gives } \frac{n!}{k!(n-k)!} \text{ samples.}$$

An unordered set of distinct objects is called a *combination*. The number of combinations of k objects from a set of n objects is denoted $C(n,k)$, or $\binom{n}{k}$. The latter notation is normally read n *choose* k. Note that

$$C(n,k) = \binom{n}{k} \quad \text{and} \quad P(n,k) = \binom{n}{k}k!.$$

If A is a set of n things, this form of counting is modeled by

$$\mathcal{P}(A)_k = \{ \text{ subsets of } A \text{ of cardinality } k \} \quad \text{where} \quad |\mathcal{P}(A)_k| = \frac{n!}{k!(n-k)!}.$$

2.5. Unordered Sampling with Replacement. This is the most difficult form of counting to understand, and we postpone discussing it for now. The formula for it is given in the chart below.

2.6. **Summary.** Our four counting techniques produce four formulae, which we summarize here. If we select k objects from a set of n objects, we recapitulate the number of ways to do this with each counting technique.

	Ordered	Unordered
With Replacement	n^k	$\binom{n+k-1}{k}$
Without Replacement	$\dfrac{n!}{(n-k)!}$	$\binom{n}{k}$